Quiver gauge theory and extended electric-magnetic duality

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ABSTRACT: We construct $\mathcal{N} = 1$ *A-D-E* quiver gauge theory with the gauge kinetic term which depends on the adjoint chiral superfields, as a low energy effective theory on D5-branes wrapped on 2-cycles of Calabi-Yau 3-fold in IIB string theory. The field-dependent gauge kinetic term can be engineered by introducing *B*-field which holomorphically varies on the base space (complex plane) of Calabi-Yau. We consider Weyl reflection on *A-D-E* node, which acts non-trivially on the gauge kinetic term. It is known that Weyl reflection is related to $\mathcal{N} = 1$ electric-magnetic duality. Therefore, the non-trivial action implies an extension of the electric-magnetic duality to the case with the field-dependent gauge kinetic term. We show that this extended duality is consistent from the field theoretical point of view. We also consider the duality map of the operators.

Contents

1 Introduction

In the previous decade, various exciting investigations have been made on 4d, $\mathcal{N} = 1$ supersymmetric gauge theory. In string theory point of view, supersymmetric gauge theory can be realized as a low energy effective theory on D-branes. The low energy behavior of supersymmetric gauge theory has been widely analyzed by using gauge/gravity correspondence. In particular, it has been known that the effective superpotential of $\mathcal{N} = 1$ supersymmetric gauge theory with an adjoint chiral superfield and a tree level superpotential can be evaluated from the gravity theory with flux [1–3] and from the matrix model [4]. These relations have been analyzed in [5–8] field-theoretically.

Recently, some interesting results have been obtained in $\mathcal{N} = 1$ supersymmetric U(N) gauge theory with the gauge kinetic term which depends on the adjoint chiral superfield, $\mathrm{Im} \int d^2\theta \, \mathrm{Tr}\, \tau(\phi)\mathcal{W}^\alpha\mathcal{W}_\alpha$. In [9, 10], it has been shown that the effective superpotential of such a theory is deformed compared to the theory with constant τ (in [9], a specific case where $\mathcal{N} = 2$ supersymmetry is spontaneously broken to $\mathcal{N} = 1$ [11, 12] has been analyzed). Since there are non-renormalizable coupling constants, this theory should have UV completion. In [13], this theory is obtained as a low energy effective theory on D5-branes wrapped on S^2's in Calabi-Yau 3-fold. The field-dependent gauge kinetic term is introduced by the integrals over S^2's of non-trivial B-field flux which holomorphically varies on the base space (complex plane) of Calabi-Yau. By using geometric transition duality, the

deformed superpotential [9, 10] has been derived from the gravity theory [13]. Also, it has been argued that the deformation of the effective superpotential leads to the existence of supersymmetry breaking vacua in some cases of the parameters. (See also [14] for IIA and M-theory perspectives and [15] for the case with partially broken $\mathcal{N} = 2$ supersymmetry.)

In this paper, we study $\mathcal{N} = 1$ supersymmetric gauge theory with the flavors where the gauge kinetic term depends on the adjoint chiral superfields. In the first half of the paper, we analyze $\mathcal{N} = 1$ A-D-E quiver gauge theories. As in [3, 16, 17], $\mathcal{N} = 1$ A-D-E quiver gauge theories can be obtained as low energy effective theories on D5-branes wrapped on S^2's in Calabi-Yau 3-folds (and D3-branes in affine case) in IIB string theory. More precisely, these Calabi-Yau 3-folds are constructed by fibering the non-singular spaces, which are obtained from the ALE spaces with A-D-E singularity, over the complex plane t. As in [13] for A_1 case, the non-trivial B-field flux which depends on t is turned on in order to obtain the field-dependent gauge kinetic term.

An interesting result of the string theory construction of A-D-E quiver gauge theories is that the theory which is obtained by Weyl reflection on a node of the Dynkin diagram (or equivalently quiver diagram) is equivalent to the original one, due to ambiguities from the fact that there is no unique way to blow up the singularity. Since the nodes of the Dynkin diagrams correspond to S^2's, Weyl reflection acts on the gauge kinetic term as well as the superpotential in the gauge theory. In [3, 18], it has been analyzed, in the constant τ case, that the theory obtained by Weyl reflection is a dual description (by Kutasov duality [19–23]), after integrating out the meson fields and the flavors. (See also [24, 25] for the case with antibranes.)

In the case which we will consider in this paper, the gauge kinetic term is affected by Weyl reflection. Therefore, we propose an extension of the Kutasov duality: $\mathcal{N} = 1$ supersymmetric $\mathrm{U}(N_c)$ gauge theory with an adjoint ϕ and N_f flavors Q and $\bar{Q}$, equipped with the gauge kinetic term where $\tau(\phi)$ is

$$\tau(\phi) = \sum_{k=0}^{m} t_k \phi^k, \tag{1.1}$$

and a superpotential, has a dual description which is $\mathrm{U}(nN_f - N_c)$ gauge theory with an adjoint $\tilde{\phi}$, N_f flavors q and $\bar{q}$ and meson fields, where the gauge kinetic term is

$$\tilde{\tau}(\tilde{\phi}) = \tilde{t}_0 - \sum_{k=1}^{m} t_k \tilde{\phi}^k \tag{1.2}$$

and the dual superpotential is the same as the one in [19, 21, 22]. In the latter half of this paper, we will analyze this duality from the field theoretical point of view.

The dual superpotential and the dual gauge kinetic term can be determined by the consistency of the duality. In particular, a strong constraint is that both theories should split to decoupled SQCD theories at low energy and there exists a corresponding SQCD theory in the dual theory which is Seiberg dual [26] to each decoupled SQCD theory in the original one. The duality map of several operators can also verified by using the above argument and the generalized Konishi anomaly equations. (See also [27] for a different analysis of this theory.)

The organization of this paper is as follows. We introduce $\mathcal{N} = 1$ *A-D-E* quiver gauge theories with the gauge kinetic terms which depend on the adjoint chiral superfields, promoting the gauge coupling constants τ_i to the field-dependent functions $\tau_i(\phi_i)$, in section 2. We will show that the classical equations of motion reduce to the same vacuum equations as those in the theories with constant τ_i. In section 3, we construct such theories in the framework of superstring theory. We then consider a duality: Weyl reflection on *A-D-E* nodes in section 4. We will see that this induces a non-trivial action on the gauge kinetic term as well as the superpotential. In section 5, we consider a non-trivial check of the duality proposal. Finally, we will analyze the duality map of the operators, in section 6.

2   *A-D-E* quiver gauge theories

In this section, we introduce $\mathcal{N} = 1$ *A-D-E* quiver gauge theories. Throughout this paper, we consider the case where the gauge kinetic term depends on the adjoint chiral superfields. This is an extended version of the quiver gauge theories considered in [3, 16, 17].

These theories are considered as a deformation of $\mathcal{N} = 2$ *A-D-E* quiver gauge theories whose gauge groups are $\prod_i \mathrm{U}(N_i)$ and each gauge factor corresponds to each node of the quiver diagrams. The quiver diagrams are expressed by the Dynkin diagrams of non-affine or affine $G = A, D, E$ groups. In terms of $\mathcal{N} = 1$ superfields, $\mathcal{N} = 2$ quiver gauge theory consists of the vector superfields V_i (or the field strength superfields $\mathcal{W}_i^\alpha$), the adjoint chiral superfields ϕ_i and the matter chiral superfields Q_{ij} and Q_{ji} which are respectively in the bi-fundamental representations $(N_i, \bar{N}_j)$ and $(\bar{N}_i, N_j)$ of $\mathrm{U}(N_i) \times \mathrm{U}(N_j)$ groups. (i label the nodes of the quiver diagram.) We consider the case where the prepotential which determines the $\mathcal{N} = 2$ classical Lagrangian has higher order terms, that is,

$$\mathcal{F}_i(\Psi_i) = \sum_{k=0} \frac{t_{i,k}}{(k+1)(k+2)} \Psi_i^{k+2}, \tag{2.1}$$

where Ψ_i are the $\mathcal{N} = 2$ vector superfields which contain ϕ_i and $\mathcal{W}_i^\alpha$ and $t_{i,k}$ are complex parameters. In $\mathcal{N} = 1$ superspace formalism, this leads to the field-dependent gauge kinetic term

$$\sum_i \mathrm{Im} \int d^2\theta \, \mathrm{Tr} \, \tau_i(\phi_i) \mathcal{W}_i^\alpha \mathcal{W}_{i\alpha}, \tag{2.2}$$

and also the Kähler terms. Here τ_i are related with the prepotentials as $2\tau_i(x) = \mathcal{F}_i''(x)$.

We add the superpotentials $W_i(\phi_i)$ which break $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$. We will choose these superpotentials to be polynomials of the same degree $n+1$, for simplicity. Therefore, the holomorphic part of the Lagrangian is

$$\sum_i \left[\mathrm{Im} \int d^2\theta \, \mathrm{Tr} \, \tau_i(\phi_i) \mathcal{W}_i^\alpha \mathcal{W}_{i\alpha} + \int d^2\theta \left(\mathrm{Tr} \sum_j s_{ij} Q_{ij} Q_{ji} \phi_i - \mathrm{Tr} \, W_i(\phi_i) \right) + h.c. \right], \tag{2.3}$$

where s_{ij} is the intersection matrix of i-th and j-th nodes, which is zero if the nodes are not linked and ± 1 if linked (and they also satisfy $s_{ij} = -s_{ji}$). In the affine case, the following

condition for the superpotentials:

$$\sum_{i=0}^{r} d_i W_i(x) = 0 \tag{2.4}$$

should be satisfied (where d_i are the Dynkin indices), if we geometrically engineer this theory [16]. In the non-affine case, there is no restriction on the superpotentials. Note that in the case where $t_{i,k} = 0$ for $k > 0$, i.e. constant τ_i case, these theories reduce to the ones analyzed in [3, 16, 17].

2.1 Classical equations of motion

The solution of the classical equations of motion in the case where $t_{i,k} = 0$, for $k > 0$ (constant τ_i case) has been derived in [16]. In non-zero $t_{i,k}$ case, the equations of motion are slightly complicated, but reduce to the same conditions as those in [16]. Let us see this in this subsection.

First of all, it is easy to see that the F-term equations are not changed compared to the case with constant τ_i. In fact, the gauge kinetic term which depends on the adjoint chiral superfields induces an additional term, $\partial_{\phi_i}\tau_i(\phi_i)$ multiplied by the fermion bilinear, in the F-term equations with respect to ϕ_i. However, the fermion does not get an expectation value in the classical vacua which we are interested in. Therefore, this term does not contribute to the solution.

On the other hand, the equations of motion with respect to D_i^a (where $a = 1, \ldots, N_i^2$ label the gauge indices of $U(N_i)$ gauge group) lead to

$$D_i^a = \frac{i}{2}(f_i)^a{}_{bc}\bar{\phi}_i^b\phi_i^c - (\mathrm{Im}\tau_i(\phi_i)^{-1})^{ab}\left[\mathrm{Tr}\sum_j s_{ij}(Q_{ij}^\dagger t_b^i Q_{ij} - Q_{ji}t_b^i Q_{ji}^\dagger) + h.c.\right] = 0, \tag{2.5}$$

where $(f_i)^a{}_{bc}$ and t_a^i are the structure constants and the generators in the fundamental of $U(N_i)$. Each matrix $(\mathrm{Im}\tau_i(\phi_i)^{-1})^{ab}$ is defined as an inverse of $N_i^2 \times N_i^2$ matrix $\mathrm{Tr}(\mathrm{Im}\tau_i(\phi_i)t_a^i t_b^i)$. While we have non-trivial factors $(\mathrm{Im}\tau_i(\phi_i)^{-1})^{ab}$ in (2.5) compared to the case with constant τ_i (in this case, the factors are proportional to δ^{ab}), (2.5) results in

$$\sum_j s_{ij}(Q_{ij}Q_{ij}^\dagger - Q_{ji}^\dagger Q_{ji}) + h.c. = 0, \tag{2.6}$$

which are the same D-term conditions as those in the constant τ_i case. This can be seen as follows. We are interested in the vacua where the scalar fields get the diagonal vev, i.e. non-Cartan parts of $\langle\phi_i\rangle$ are zero. Under these, the first term of (2.5) is zero. Since $\det(\mathrm{Im}\tau_i(\phi_i)^{-1})^{ab} \neq 0$, the solution is trivial and we obtain (2.6).

We have shown that the vacuum conditions following from the equations of motion are the same as those in [16]. Therefore, the structure of the classical vacua is also same. So, we only explain these here.

For the non-affine case, the solutions of these equations are specified in terms of the positive roots ρ_K [16], where $K = 1, \ldots, R_+$ with $2R_+ + r = |G|$ and r is the rank of G.

In terms of the simple roots e_i, the positive roots can be written as

$$\rho_K = \sum_{i=1}^{r} n_K^i e_i, \tag{2.7}$$

where n_K^i are some positive integers. The equations of motion reduce to the following equations

$$W_K'(x) = \sum_i n_K^i W_i'(x) = 0. \tag{2.8}$$

Each of these equations has n roots for each positive root ρ_K because we have chosen every superpotential is polynomial of degree $n+1$. We denote these roots as $x = a_{(p,K)}$ where $p = 1, \ldots, n$. Then, a supersymmetric vacuum is given by the value of $a_{(p,K)}$ with multiplicities $M_{(p,K)} \geq 0$ which satisfy

$$N_i = \sum_{K=1}^{R_+} \sum_{p=1}^{n} M_{(p,K)} n_K^i. \tag{2.9}$$

Furthermore, the gauge group is broken as

$$\sum_i \mathrm{U}(N_i) \rightarrow \sum_K \sum_p \mathrm{U}(M_{(p,K)}), \tag{2.10}$$

by Higgsing.

For the affine case, we have one additional node in quiver diagram and the gauge group is $\sum_{i=0}^{r} \mathrm{U}(N_i)$ where $\mathrm{U}(N_0)$ gauge group corresponds to that node. The classical vacua are similarly specified by the positive roots as above [16].

3 Geometric construction

The above gauge theories can be realized as low energy effective theories on D5-branes wrapped on 2-cycles of Calabi-Yau 3-folds in the non-affine case and additional D3-branes in the affine case. These Calabi-Yau 3-folds are constructed by non-trivially fibering the ALE spaces with A-D-E singularity over the complex plane. The important difference between the quiver gauge theory constructed above and the one in [3, 16] is the gauge kinetic term. As considered in [13] for A_1 case, the field dependent gauge kinetic term can be engineered geometrically by introducing the non-trivial B-field depending on the complex plane which is the base space of Calabi-Yau 3-fold.

We consider the ALE spaces with A-D-E singularity at the origin, which can be viewed as the hypersurfaces $f(x, y, z) = 0$ with, e.g. for A_r singularity,

$$f = x^2 + y^2 + z^{r+1}, \tag{3.1}$$

where $x, y, z \in \mathbf{C}$. By deforming these by relevant deformations, we obtain non-singular spaces, whose defining equations are, in A_r case,

$$f = x^2 + y^2 + \prod_{i=1}^{r+1} (z + t_i), \quad \sum_{i=1}^{r+1} t_i = 0, \tag{3.2}$$

where t_i are deformation parameters and there are r independent classes of non-vanishing S^2's. These classes intersect according to the corresponding A-D-E Dynkin diagrams. The holomorphic volumes of S^2's are defined by integrals of the holomorphic 2-form $\omega = dxdy/z$ as $\alpha_i = \int_{S_i^2} \omega$. These α_i are simply related to t_i in (3.2) by, in A_r case,

$$\alpha_i = t_i - t_{i+1}. \tag{3.3}$$

For $G = D, E$ cases, the constructions are similar to the above [16].

We consider the fibrations of these spaces over the complex plane. We denote the coordinate of this plane as t. If there exists D5-branes wrapping on the above S_i^2 and occupying the $R^{1,3}$ direction, we obtain 4d non-affine A-D-E quiver gauge theories as low energy effective theories on the D5-branes, whose field contents are the same as those in section 2. If we include the D3-branes occupying $R^{1,3}$ direction, the gauge theory becomes the affine quiver gauge theory [3, 16, 28]. Note that since t plane is orthogonal to the S^2's on which D5-branes wrap, it parametrizes the positions of the D-branes. Thus, t corresponds to the vacuum expectation value of the adjoint chiral superfield on the D-brane world volume.

In type IIB string theory, there are NSNS field B^{NS} and RR field B^R. Geometrically, the complexified gauge coupling of each gauge factor of the quiver gauge theory corresponds to the integral over corresponding S_i^2 of Calabi-Yau 3-fold:

$$\left(\frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2} \right)_i = \int_{S_i^2} \left(B^R + \frac{i}{g_s} B^{NS} \right). \tag{3.4}$$

Note that we have set the Kähler parameters to zero: $r_i \equiv \int_{S_i^2} k = 0$ where k is the Kähler form. As in [3, 16], if the background B-fields do not have any t-dependence, the above quantities are constants and denote the complexified gauge coupling constants. On the other hand, if the background B-fields depend on t as in [13], we obtain

$$\tau_i(t) \equiv \left(\frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2} \right)_i (t) = \int_{S_i^2} \left(B^R(t) + \frac{i}{g_s} B^{NS}(t) \right), \tag{3.5}$$

which produce the field-dependent gauge kinetic term of the effective theory on the D-branes, as in section 2. The point is that in order not to break the $\mathcal{N} = 2$ supersymmetry, B-fields should *holomorphically* depend on t [29]. Indeed, the dual IIB supergravity solution of this brane set-up, which has $\mathcal{N} = 2$ supersymmetry in 4d, can be obtained assuming that the dilaton is constant. Generically, t-dependent B-fields induce the source term in the dilaton equation of motion. However, such a source term vanishes in the case with holomorphically t-dependent B-fields [29]. Therefore, the dilaton remains constant in this case and $\mathcal{N} = 2$ supersymmetry is not broken.

The superpotentials can be turned on by considering the non-trivial fibration of the ALE space over t-plane, promoting α_i (3.3) to be dependent on t: $\alpha_i = \alpha_i(t)$. These α_i give the superpotential $W_i'(z) = \alpha_i(z)$. We only consider the non-monodromic fibration where α_i are the single-valued functions of t as it leads to the single trace functions $W_i(\phi_i)$ in section 2. Also, we choose all the degrees of the superpotentials to be $n + 1$. In this

case, there are n points in t plane for each positive root ρ_K (2.7) where the holomorphic volume becomes zero

$$W'_K(t) \equiv \sum_{i=1}^{r} n_K^i W'_i(t) = 0. \tag{3.6}$$

These equations correspond to the conditions for the supersymmetric vacua and are same as those obtained in the gauge theory (2.8). The roots of (3.6) are expressed as $t = a_{(p,K)}$ where $p = 1, \ldots n$. As noted above, these values correspond to the positions of D-branes and, therefore, the vacuum expectation values of ϕ_i.

Therefore, we have geometrically engineered the $\mathcal{N} = 1$ A-D-E quiver gauge theories with the field-dependent gauge kinetic term, which have been considered in the previous section. This construction is a simple generalization of A_1 case [13] to other quiver cases.

Now we will comment on an important point which arises from the non-trivial fields background. Note that the background B-fields (3.5) and the positions of D-branes $a_{(p,K)}$ determine the classical gauge coupling constant of each gauge factor $\mathrm{U}(M_{(p,K)})$

$$\left(\frac{4\pi}{g^2}\right)_{(p,K)} = \sum_{i=1}^{r} n_K^i \mathrm{Im}\tau_i(a_{(p,K)}). \tag{3.7}$$

The crucial point is that, in contrast to the case in [3], these quantities could be negative for generic choice of the background fields and the superpotentials. This implies that the field theoretical description is ill-defined in that case, but from the string theory point of view, the case where some of the squared gauge coupling constants are negative arises from antibranes wrapping on the corresponding S^2's.[1]

4 Duality in string theory

As considered in [3], there are two types of duality in the above theories. The one is the geometric transition duality [1] and the other one corresponds to Weyl reflection of A-D-E groups. In this paper, we only consider the latter type.

Weyl reflection about the simple root e_{i_0} of A-D-E group can be viewed as the following action on the simple roots:

$$e_i \rightarrow e_i - (e_i \cdot e_{i_0})e_{i_0}, \tag{4.1}$$

where the inner product of the simple roots is normalized as follows: $e_i \cdot e_{i_0}$ are 2 for $i = i_0$, -1 for i connected with i_0 node and 0 for the other i. In the Calabi-Yau geometry, this corresponds to the change of S^2's and leads to the following action on τ_i and the polynomial parts of the superpotential:

$$\tau_i(\phi_i) \rightarrow \tau_i(\phi_i) - (e_i \cdot e_{i_0})\tau_{i_0}(\phi_i), \quad W_i(\phi_i) \rightarrow W_i(\phi_i) - (e_i \cdot e_{i_0})W_{i_0}(\phi_i). \tag{4.2}$$

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[1] In A_1 case, if all the squared gauge coupling constants are negative, a better field theoretical description which is supersymmetry breaking model by spurion fields [30, 31] has been proposed in [13].

The action of the Weyl reflection on the superpotentials are exactly same as those in [3]. But, since the gauge couplings τ_i are polynomials of ϕ_i, the higher order terms in ϕ_i are also affected by the Weyl reflection. This induces non-trivial action on the coupling constants $t_{i,k}$ in τ_i. In fact, in the case with constant τ_i, this reduces to the action on the gauge coupling constants, as in [3]: $(1/g^2)_i \to (1/g^2)_i - (e_i \cdot e_{i_0})(1/g^2)_{i_0}$.

The different looking gauge theory obtained by Weyl reflection should be equivalent to the original one from the string theory perspective [3]. This is due to ambiguities which come from the fact that there is no unique way to blow up the singularity and we can determine a quiver gauge theory up to Weyl group action. Since the total brane charge must be conserved, the ranks of the gauge groups after the transition are related with the original ranks as

$$\sum_i N_i' e_i' = \sum_i N_i e_i. \tag{4.3}$$

Hence, the ranks of the gauge groups are changed under the Weyl reflection about e_{i_0} as $N_{i_0}' = N_f - N_{i_0}$ and $N_i' = N_i$ for $i \neq i_0$ where N_f is the number of flavors of $\mathrm{U}(N_{i_0})$ gauge theory when the other gauge symmetries are considered as flavor symmetries and $N_f \equiv \sum_{i \neq i_0} (-e_i \cdot e_{i_0}) N_i$. Note that the number of flavors is not changed under the Weyl reflection.

As discussed in [3], this kind of duality can be considered as $\mathcal{N} = 1$ electric-magnetic duality [19–22, 26, 36] in the framework of the gauge theory. (See also [32–35] for related approaches.) However, as seen above, the duality induces the non-trivial action on the field-dependent gauge kinetic term. This is a first example for the electric-magnetic duality in the case with the field dependent gauge kinetic term. We will call this as extended electric-magnetic duality. Fortunately, string theory has suggested that such a duality exists. We will check this duality field-theoretically in the subsequent sections.

Before going to next, let us see the action on the superpotentials and on τ_i more explicitly. First of all, the action on the superpotentials (4.2) can be written as

$$W_i'(\phi_i) = \begin{cases} -W_i(\phi_i), & \text{for } i = i_0, \\ W_i(\phi_i) + W_{i_0}(\phi_i), & \text{for } i \text{ connected with } i_0, \\ W_i(\phi_i), & \text{for the other } i. \end{cases} \tag{4.4}$$

Also, for the coefficients of the gauge kinetic terms, Weyl reflection acts as

$$\tau_i'(\phi_i) = \begin{cases} -\tau_i(\phi_i), & \text{for } i = i_0, \\ \tau_i(\phi_i) + \tau_{i_0}(\phi_i), & \text{for } i \text{ connected with } i_0, \\ \tau_i(\phi_i), & \text{for the other } i, \end{cases} \tag{4.5}$$

Let us concentrate on the gauge theory on the i_0-th node. If we treat the gauge symmetries of the linked nodes as the weakly gauged flavor symmetries, we obtain $\mathrm{U}(N_{i_0})$ gauge theory with a superpotential

$$W = \sum_{k=1}^{n} \frac{g_k}{k+1} \operatorname{Tr} \phi^{k+1} + \operatorname{tr} \bar{Q} \phi Q + \operatorname{tr} m \bar{Q} Q, \tag{4.6}$$

where Q and $\bar{Q}$ are N_f fundamental and anti-fundamental superfields. The symbol tr denotes the trace over the flavor indices. These come from the bi-fundamental superfield connecting i_0-th node with the neighboring nodes. The mass term for Q and $\bar{Q}$ is due to $\phi_{i'}Q_{i'i_0}Q_{i_0i'}$ term of the neighboring nodes by giving a vev of $\phi_{i'}$. Also, let the gauge kinetic term of this theory be

$$\mathrm{Tr}\,\tau_{i_0}(\phi)\mathcal{W}^\alpha\mathcal{W}_\alpha = \sum_{k=0}^n t_k\,\mathrm{Tr}(\phi^k\mathcal{W}^\alpha\mathcal{W}_\alpha), \tag{4.7}$$

where we have simplified the notation of the coupling in τ_{i_0} as $t_{i_0,k} \equiv t_k$.

The Weyl action changes the gauge group to $\mathrm{U}(N_f - N_{i_0})$ $(N_f \equiv \sum_{i \neq i_0}(-e_i \cdot e_{i_0})N_i)$ and the superpotential (4.6) to

$$\widetilde{W} = -\sum_{k=1}^n \frac{g_k}{k+1}\,\mathrm{Tr}\,\tilde{\phi}^{k+1} + \sum_{k=1}^n \frac{g_k}{k+1}\mathrm{tr}m^{k+1} + \mathrm{tr}\bar{q}\tilde{\phi}q + \mathrm{tr}m\bar{q}q, \tag{4.8}$$

where $\tilde{\phi}$ is an adjoint field of $\mathrm{U}(N_f - N_{i_0})$ gauge group and q and $\bar{q}$ are the N_f fundamentals and anti-fundamentals. The minus sign of the first term reflects the Weyl action on the superpotential W_{i_0} (4.4). The second term comes from the Weyl action on the superpotentials of the nodes linked to i_0 node. (The trace of this term is taken over the flavor indices.) Furthermore, the gauge kinetic term of the dual theory becomes

$$-\sum_{k=0}^n t_k\,\mathrm{Tr}(\tilde{\phi}^k\widetilde{\mathcal{W}}^\alpha\widetilde{\mathcal{W}}_\alpha), \tag{4.9}$$

where $\widetilde{\mathcal{W}}^\alpha$ is the field strength superfield of the dual theory.

In the theory corresponding to a node connected with i_0-th node, as noted above, the dual superpotential of this theory contributes to the second term of (4.8) because the Weyl reflection induces an additional term $W_{i_0}(\tilde{\phi}_i)$ (4.4) where $\tilde{\phi}_i$ is an chiral superfield of this dual theory. On the other hand, the gauge kinetic term is affected as follows:

$$\tau_i(\phi_i) \to \tau_i(\tilde{\phi}_i) + \tau_{i_0}(\tilde{\phi}_i) = \tau_i(\tilde{\phi}_i) + \sum_{k=0}^n t_k\tilde{\phi}_i^k, \tag{4.10}$$

as easily extracted from (4.5).

5   Extended electric-magnetic duality

We have seen that the string theory construction has suggested an extension of $\mathcal{N} = 1$ electric-magnetic duality to the case where the gauge kinetic term depends on the adjoint chiral superfields. In what follows, we concentrate on a particular node of the quiver and consider the duality from field-theoretical point of view.

Let us specify the model. Consider $\mathcal{N} = 1$, $\mathrm{U}(N_c)$ gauge theory with an adjoint chiral superfield ϕ and N_f fundamental and anti-fundamental superfields Q and $\bar{Q}$, and also with a gauge kinetic term which depends on the adjoint chiral superfield:

$$\int d^2\theta\,\mathrm{Tr}\,\tau(\phi)\mathcal{W}^\alpha\mathcal{W}_\alpha, \quad \tau(\phi) = \sum_{k=0}^m t_k\phi^k, \tag{5.1}$$

and a superpotential (4.6)

$$W = \sum_{k=1}^{n} \frac{g_k}{k+1} \operatorname{Tr} \phi^{k+1} + \operatorname{tr} \bar{Q} \phi Q + \operatorname{tr} m \bar{Q} Q. \tag{5.2}$$

Without loss of the generality, the mass matrix m can be chosen to be diagonal. We will use the indices $f = 1, \ldots, N_f$ to label the flavors. In this notation, the diagonal components of the mass matrix are written as m_f.

The vacua of this theory can be divided into two types: confining and Higgs vacua. Classically, the confining vacua correspond to the vacuum expectation values such that $\langle Q \rangle = \langle \bar{Q} \rangle = 0$ and

$$\langle \phi \rangle = \operatorname{diag}(a_1, a_2, \ldots, a_N), \tag{5.3}$$

where a_i are determined from the solutions of the F-term equation:

$$W'(x) \equiv g_n \prod_{i=1}^{n} (x - a_i) = 0. \tag{5.4}$$

Note that the other terms contributing to the F-term equation vanish in these vacua. Indeed, as we have seen in the quiver case, the gauge kinetic term, $\tau(\phi) \mathcal{W}^\alpha \mathcal{W}_\alpha$, produces an additional term in the F-term equation such as $\partial_\phi \tau(\phi) \lambda^\alpha \lambda_\alpha$ where λ^α is the gluino, but this term vanishes because we are interested in the vacua where the vacuum expectation values of the fermions are zero.

The Higgs vacua correspond to the case where some of the diagonal elements of $\langle \phi \rangle$ are equal to the mass parameters and Q and $\bar{Q}$ have non-zero vacuum expectation values which are determined from the F-term equation:

$$(W'(\phi))_{ij} + \sum_f Q_j^f \bar{Q}_{if} = 0, \tag{5.5}$$

where $i, j = 1, \ldots, N$ are the gauge indices. As above, the gauge kinetic term does not contribute to the classical equation (5.5).

In subsection 5.1, we begin to consider the case without $\bar{Q} \phi Q$ and $m \bar{Q} Q$ terms. In this case, the flavors are massless and, after integrating out the adjoint fields, the theory splits into a set of the decoupled SQCD theories with the massless flavors. Therefore, the stable vacua exist if [21]

$$\frac{N_c}{n} \le N_f. \tag{5.6}$$

We will see the dual description of the above theory, after reviewing the constant τ case. Then, we will turn to the case with full superpotential (5.2) in subsection 5.2.

5.1 Single trace superpotential case

We consider the case where the superpotential is

$$W = \sum_{k=1}^{n} \frac{g_k}{k+1} \operatorname{Tr} \phi^{k+1}. \tag{5.7}$$

	ϕ	$\mathcal{W}^\alpha$	Q (or $\bar{Q}$)	g_k	$\Lambda^{2N_c - N_f}$
$\mathrm{U}(1)_R$	2	1	0	$-2k$	$2(2N_c - N_f)$
$\mathrm{U}(1)_J$	0	1	1	2	0

Table 1. U(1) charges of the electric fields and the parameters.

	$\tilde{\phi}$	$\widetilde{\mathcal{W}^\alpha}$	q (or $\bar{q}$)	M_ℓ	g_k	μ	$\tilde{\Lambda}^{2\tilde{N}_c - N_f}$
$\mathrm{U}(1)_R$	2	1	$\frac{(n-1)(N_c-\tilde{N}_c)}{\tilde{N}_c}$	$2(\ell-1)$	$-2k$	$-2+\frac{(n-1)(N_c-\tilde{N}_c)}{\tilde{N}_c}$	$4\tilde{N}_c - 2N_f+\frac{2N_f(n-1)(N_c-\tilde{N}_c)}{\tilde{N}_c}$
$\mathrm{U}(1)_J$	0	1	$\frac{N_c}{\tilde{N}_c}$	2	2	$1+\frac{N_c}{\tilde{N}_c}$	$\frac{2N_f(N_c-\tilde{N}_c)}{\tilde{N}_c}$

Table 2. U(1) charges of the magnetic fields and the parameters.

We first review the case where τ is constant, i.e. $t_k = 0$ for $k > 0$. In this case, the dual description of the theory has been obtained in [19, 21, 22], which is $\mathrm{U}(\tilde{N}_c)$ ($\tilde{N}_c = nN_f - N_c$) gauge theory with N_f fundamental and anti-fundamental superfields q and $\bar{q}$, gauge singlet superfields M_i ($i = 1, \ldots, n$) and an adjoint chiral superfield $\tilde{\phi}$. The singlet fields M_i are identified with the meson superfields in the original theory as

$$M_i = \bar{Q}\phi^{i-1}Q, \quad i = 1, \ldots, n. \tag{5.8}$$

It is not necessary to introduce the other meson fields corresponding to $\bar{Q}\phi^\ell Q$ ($\ell > n$), since such fields can be eliminated by the chiral ring relation. In addition, the superpotential of the dual theory is [22]

$$\widetilde{W} = -\sum_{k=1}^{n} \frac{g_k}{k+1} \operatorname{Tr} \tilde{\phi}^{k+1} + \frac{1}{\mu^2} \sum_{k=1}^{n} g_k \sum_{i=1}^{k} M_i \bar{q} \tilde{\phi}^{k-i} q, \tag{5.9}$$

where a parameter μ has been introduced in order for the dimension of the second term to be correct. This duality is a generalization of the electric-magnetic duality (Seiberg duality) in $\mathcal{N} = 1$ SQCD [26] to the case with an adjoint chiral superfield and a tree level single trace superpotential. Below we refer to the original and dual theories as electric and magnetic theories respectively.

The global symmetries of both theories are the same: there is $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f) \times \mathrm{U}(1)_R \times \mathrm{U}(1)_J$ symmetry. The U(1) charges of the fields (and the parameters) of the electric theory are in table 1, where Λ is the dynamical scale of the electric theory and the superspace coordinate θ also has charge 1. Note that we have allowed the coupling constants g_k and t_k transform non-trivially, as $\mathrm{U}(1)_R$ and $\mathrm{U}(1)_J$ become the symmetries with the superpotential and the gauge kinetic term. Also, the charges of the fields of the magnetic theory are in table 2, where $\tilde{\Lambda}$ is the dynamical scale of the magnetic theory. A non-trivial check of this duality is to compare the 't Hooft anomalies of the theories. It has been shown that they perfectly match in the case with the truncated superpotential [21].

An important ingredient of $\mathcal{N}=1$ duality is the matching relation of the dynamical scales. In the case here, the relation is

$$\Lambda^{2N_c - N_f} \tilde{\Lambda}^{2\tilde{N}_c - N_f} = g_n^{-2N_f} \mu^{2N_f}. \tag{5.10}$$

One can easily check that this relation is consistent with the above charge assignment. Also, in [22], it has been shown that this is consistent with the deformations of the theory by the mass terms of the flavors.

It is worth noting that, on general grounds, the coefficients of $\text{Tr}\,\tilde{\phi}^{k+1}$ and $M_i \bar{q} \tilde{\phi}^{k-i} q$ in the magnetic superpotential are generic functions of g_k. However, we can fix these coefficients as in (5.9). First of all, in the electric theory, according to (5.3), the gauge symmetry is broken to $\prod_{i=1}^{n} \text{U}(r_i)$ where $\sum_i r_i = N_c$. (r_i denote the number of the eigenvalues of $\langle \phi \rangle$ which are equal to a_i.) Supposing that the underlying $\text{U}(N_c)$ gauge theory is weakly coupled at the mass scale which is specified by the above superpotential, the theory splits in the low energy into a set of decoupled SQCD theories with $\text{U}(r_i)$ gauge groups and N_f flavors.[2] In the dual theory, the coefficients of $\text{Tr}\,\tilde{\phi}^{k+1}$ have been fixed such that the magnetic superpotential has the same critical points a_i as those in the electric theory. Then, we observe a similar gauge symmetry breaking pattern: $\text{U}(nN_f - N_c) \rightarrow \prod_i \text{U}(\tilde{r}_i)$. The claim is that $\tilde{r}_i = N_f - r_i$,[3] in order to obtain one-to-one correspondence between each $\text{U}(\tilde{r}_i)$ SQCD theory and each of the decoupled SQCD theories in the electric theory under Seiberg duality [26].[4] Also, the magnetic theory should split to $\text{U}(N_f - r_i)$ SQCD theories with N_f flavors and mesons, as the electric theory does. This determines the coefficients of $M_i \bar{q} \tilde{\phi}^{k-i} q$ and leads to (5.9) [22].

Magnetic gauge kinetic term. We now turn to the analysis of the gauge kinetic term. We first note that inclusion of the ϕ-dependent part of τ does not change the structure of the classical chiral ring. The classical chiral ring relations, i.e. a set of constraints on the gauge invariant operators follows from the F-term equation (and a constraint on characteristic polynomial: $f(\phi) = 0$ with $f(x) = \det(x - \phi)$). Indeed, as we have seen above, $\tau(\phi) \mathcal{W}^\alpha \mathcal{W}_\alpha$ term does not affect the classical solution. On the other hand, the quantum chiral ring is modified by the existence of the field dependent part of τ because the gluino confines in the confining vacua and leads to the non-zero vacuum expectation value of $\langle \lambda^\alpha \lambda_\alpha \rangle$. We will see this in next section by analyzing the generalized Konishi anomaly equations.

Now, consider the magnetic superpotential. In general, it could depend on t_k as well as g_k. Recall however that the magnetic superpotential has been determined such that it has the same critical points as those of the electric theory and it is consistent with the decoupling of the SQCD theories in the magnetic theory. This process can be applied to the case with the field-dependent gauge kinetic term: if the magnetic superpotential depends on t_k, we can no longer obtain the same critical points. Also, t_k-dependent $M_i \bar{q} \tilde{\phi}^{k-i} q$ terms

---

[2]This is the case where all the roots of (5.4) are different from each other. In the case where some of a_i coincide, i.e. $W' = \prod_{i=1}^{r}(x - a_i)^{n_i}$ $(r < n)$, each decoupled theory has a superpotential as $\text{Tr}\,\phi_i^{n_i}$.

[3]In the case corresponding to the above footnote, the corresponding gauge group is $\text{U}(n_i N_f - r_i)$.

[4]In the case corresponding to the above footnotes, we demand that each decoupled theory in the magnetic theory is related with each decoupled theory in the electric theory by Kutasov duality [19, 21] with the truncated superpotential

obviously make decoupled SQCD theories to couple each other. Therefore, the magnetic superpotential cannot depend on t_k.

On the other hand, the gauge kinetic term of the magnetic theory can be written generally as

$$\sum_{k=0}^{n} \tilde{t}_k \operatorname{Tr} \tilde{\phi}^k \widetilde{\mathcal{W}}^\alpha \widetilde{\mathcal{W}}_\alpha, \tag{5.11}$$

where $\tilde{t}_k$ are some functions of the parameters in the electric theory, which relate the coupling constants of the electric theory with those of the magnetic theory. As the coupling constants in the magnetic superpotential have been fixed such that the magnetic theory correctly behaves as the dual of the original one, we have to choose the correct form of the functions $\tilde{t}_k(t,g)$. We will see below that this is simply

$$\tilde{t}_k = -t_k, \tag{5.12}$$

for $k = 1, \ldots, m$. The lowest coupling constant, i.e. $\tilde{t}_0$, can also be determined from the matching relation of the dynamical scales (5.13).

Let us see (5.12) is indeed the case. We first consider the matching relation of the dynamical scales of the electric and magnetic theories. As we have seen above, the matching relation in the case with constant τ is (5.8)

$$\Lambda^{2N_c - N_f} \tilde{\Lambda}^{2\tilde{N}_c - N_f} = g_n^{-2N_f} \mu^{2N_f}. \tag{5.13}$$

In the case with $\tau(\phi)$, we can assign $\mathrm{U}(1)_R$ and $\mathrm{U}(1)_J$ charges to t_k and $\tilde{t}_k$ as

	t_k	$\tilde{t}_k$
$\mathrm{U}(1)_R$	$-2k$	$-2k$
$\mathrm{U}(1)_J$	0	0

in addition to the charge assignment in table 1 and 2. It follows from the above global U(1) charges and also the consistency with the mass (of the flavor) deformation as in [22] that this relation cannot change even if we add the parameters t_k and $\tilde{t}_k$ to the theory. Therefore, the relation is valid in the case we consider here.

By integrating the massive vector superfields and the massive adjoint field out in both theories, the matching relation leads to

$$\Lambda_i^{3r_i - N_f} \tilde{\Lambda}_i^{3\tilde{r}_i - N_f} = (-)^{N_f - r_i} g_n^{-N_f} \mu^{2N_f} e^{-2\pi i (T(a_i) + \tilde{T}(a_i))} \prod_{j \neq i} (a_i - a_j)^{-N_f}, \tag{5.14}$$

for each i. We have defined as $T(x) = \tau(x) - t_0$ and $\tilde{T}(x) = \tilde{\tau}(x) - \tilde{t}_0$. Λ_i and $\tilde{\Lambda}_i$ are dynamical scales of $\mathrm{U}(r_i)$ and $\mathrm{U}(\tilde{r}_i)$ theories ($\tilde{r}_i \equiv N_f - r_i$), which are defined by the matching of the gauge coupling constants:

$$\Lambda^{2N_c - N_f} = \Lambda_i^{3r_i - N_f} \frac{e^{2\pi i T(a_i)}}{(W''(a_i))^{r_i}} \prod_{j \neq i} (a_i - a_j)^{2r_j}, \tag{5.15}$$

and the similar equations for the magnetic variables. In (5.15), $\prod_{j \neq i}(a_i - a_j)^{2r_j}$ factor comes from the integration of the massive vector superfields and $(W''(a_i))^{r_i}$ factor is due to the massive adjoint field. Furthermore, we add the factor $e^{2\pi i T(a_i)}$ because the gauge kinetic term depends on the adjoint field.

Finally, we note that the relation (5.14) should be consistent with the decoupling of the SQCD theories in the electric and magnetic theories at low energy. This implies that the following relations

$$\Lambda_i^{3r_i - N_f} \tilde{\Lambda}_i^{3\tilde{r}_i - N_f} = (-)^{N_f - r_i} \mu_i^{N_f} \tag{5.16}$$

are satisfied for each decoupled SQCD [26], where μ_i are the parameters in the magnetic superpotentials of $U(\tilde{r}_i)$ SQCD theories, $\mu_i^{-1} \bar{q}_i q_i M_i$. Since we can show that $g_n^{-1} \mu^2 \prod_{i \neq j}(a_i - a_j)^{-1} = \mu_i$ as in [22], we therefore obtain

$$\sum_{k=1}^m t_k(a_i)^k = -\sum_{k=1}^m \tilde{t}_k(a_i)^k, \tag{5.17}$$

which implies $\tilde{t}_k = -t_k$ for $k = 1, \ldots, m$. In principle, (5.17) could have an additional integer term. However, such a term must vanish since there is no way to satisfy the equality with that term. Note that the parameters t_k are the values at the energy scale where the gauge symmetry is broken.

Note also that the argument above is valid only in the region where the gauge coupling constant is small. We will see in section 6.2 that (5.12) can be verified by using a different method.

5.2 Generic superpotential case

Based on the above argument, let us consider the case with more generic superpotential which has been appeared in the string theory construction:

$$W = \sum_{k=1}^n \frac{g_k}{k+1} \operatorname{Tr} \phi^{k+1} + \operatorname{tr}\bar{Q}\phi Q + \operatorname{tr}m\bar{Q}Q. \tag{5.18}$$

As discussed in [20, 23], by flowing from the theory considered in the previous subsection or in [22], we can deduce that the dual superpotential becomes

$$\widetilde{W} = -\sum_{k=1}^n \frac{g_k}{k+1} \operatorname{Tr} \tilde{\phi}^{k+1} + \frac{1}{\mu^2} \sum_{k=1}^n g_k \sum_{i=1}^k M_i \bar{q} \tilde{\phi}^{k-i} q + \lambda M_2 + m M_1. \tag{5.19}$$

By the relations (5.9), the last two terms correspond $\bar{Q}\phi Q$ and the mass deformations. What we have to check about this superpotential is whether the deformation terms do not spoil the separation of the SQCD theories or not, as we have discussed in the previous subsection. But it is obviously trivial since the last two terms have no room to mix the operators of the different gauge factors.

The analysis of the dual gauge kinetic term is the same as that of the previous subsection and we do not repeat here. The conclusion is $\tilde{t}_k = -t_k$. This is exactly same as what has been expected in the string theory (4.9).

While we have formulated a magnetic dual, the magnetic superpotential (5.19) is different from the one expected from the string theory duality (4.8). In fact, the dual theory obtained by Weyl reflection in string theory does not include the meson fields and the gauge groups are also different: $U(N_f - N_c)$ in the stringy dual theory, and $U(nN_f - N_c)$ in the magnetic theory in present section. However, one can show that the magnetic theory reduces to the stringy dual one after integrating out the mesons and (anti-)fundamentals and Higgsing to $U(N_f - N_c)$ gauge theory as in [3]. As we have already seen, the gauge kinetic term which depends on the adjoint chiral superfield does not affect the classical equations of motion. Therefore, the discussion is the same as that in the theory with constant τ.

6 Duality map of the chiral operators

In this section, let us consider the duality map between the chiral operators in the electric theory and the magnetic ones. First of all, we consider the operators $\mathrm{Tr}\,\mathcal{W}^\alpha \mathcal{W}_\alpha$ (and $\mathrm{Tr}\,\widetilde{\mathcal{W}}^\alpha \widetilde{\mathcal{W}}_\alpha$). As already seen above, the matching relations of the dynamical scales of the decoupled SQCD theories are

$$\Lambda_i^{3r_i - N_f} \tilde{\Lambda}_i^{3\tilde{r}_i - N_f} = (-)^{N_f - r_i} \mu_i^{N_f}, \tag{6.1}$$

for each i. In each $U(r_i)$ SQCD theory, the gauge coupling constant receives one-loop correction and the gauge kinetic term is renormalized as $(3r_i - N_f)\log(\Lambda_i/M)\,\mathrm{Tr}\,\mathcal{W}^{i\alpha}\mathcal{W}_\alpha^i$ in the electric theory and $(3\tilde{r}_i - N_f)\log(\tilde{\Lambda}_i/M)\,\mathrm{Tr}\,\widetilde{\mathcal{W}}^{i\alpha}\widetilde{\mathcal{W}}_\alpha^i$ in the magnetic theory. If we take a derivative with respect to $\log \Lambda_i$ and use (5.14) as in [22, 36], we obtain the following relations:

$$\mathrm{Tr}_{U(r_i)}\mathcal{W}^{i\alpha}\mathcal{W}_\alpha^i = -\mathrm{Tr}_{U(\tilde{r}_i)}\widetilde{\mathcal{W}}^{i\alpha}\widetilde{\mathcal{W}}_\alpha^i, \tag{6.2}$$

for each i. These imply that the gauge coupling constant of each decoupled SQCD theory in the electric theory is different by sign from the magnetic one.

To check the other relations in terms of more complicated operators, it is convenient to use the generalized Konishi anomaly equations, as in [37] for the constant τ case. Thus, we first derive these equations in subsection 6.1. Then, we will consider the duality map of the operators in subsection 6.2 and 6.3.

6.1 Generalized Konishi anomaly equations

Let us derive the generalized Konishi anomaly equations in the electric and magnetic theories. We define the generating functions of the one-point functions in the electric theory as

$$
\begin{aligned}
R(z) &= -\frac{1}{64\pi^2}\left\langle \mathrm{Tr}\,\frac{\mathcal{W}^\alpha \mathcal{W}_\alpha}{z - \phi}\right\rangle, \\
T(z) &= \left\langle \mathrm{Tr}\,\frac{1}{z - \phi}\right\rangle, \\
M(z)_f^{f'} &= \left\langle \bar{Q}_f \frac{1}{z - \phi} Q^{f'}\right\rangle,
\end{aligned}
\tag{6.3}
$$

where we have ignored the fermionic one-point function. The generalized Konishi anomaly equations in terms of these variables are:[5]

$$R(z)^2 = \left[W'(z)R(z)\right]_-,$$

$$2R(z)T(z) = \left[W'(z)T(z)\right]_- + 32\pi^2 i\left[\tau'(z)R(z)\right]_- + M(z),$$

$$-\delta_f^{f'} R(z) = \left[M(z)_f^{f'}(z + m_{f'})\right]_-, \tag{6.4}$$

which can be obtained by generalizing the arguments in [6–10]. In the last equation, the flavor index f' is not contracted.

In the magnetic theory, we can also define

$$\widetilde{R}(z) = -\frac{1}{64\pi^2}\left\langle \mathrm{Tr}\,\frac{\widetilde{\mathcal{W}}^\alpha \widetilde{\mathcal{W}}_\alpha}{z - \tilde{\phi}} \right\rangle,$$

$$\widetilde{T}(z) = \left\langle \mathrm{Tr}\,\frac{1}{z - \tilde{\phi}} \right\rangle,$$

$$\widetilde{M}(z)_f^{f'} = \left\langle \bar{q}_f \frac{1}{z - \tilde{\phi}} q^{f'} \right\rangle. \tag{6.5}$$

In terms of these, the anomaly equations can be obtained as

$$\widetilde{R}(z)^2 = -\left[W'(z)\widetilde{R}(z)\right]_-,$$

$$2\widetilde{R}(z)\widetilde{T}(z) = -\left[W'(z)\widetilde{T}(z)\right]_- - 32\pi^2 i\left[\tau'(z)\widetilde{R}(z)\right]_- + \left[\widetilde{M}(z)A'(z)\right]_-,$$

$$-\delta_f^{f'}\widetilde{R}(z) = \left[\widetilde{M}(z)_f^{f''} A(z)_{f''}^{f'}\right]_-, \tag{6.6}$$

where

$$A(z) = \frac{1}{\mu^2}\sum_{k=1}^n g_k \sum_{i=1}^k M_i z^{k-i}. \tag{6.7}$$

Note that $M_i\bar{q}\phi^{k-i}q$ terms in the magnetic superpotential do not contribute to the first equation of (6.6) because the terms with $\bar{q}\mathcal{W}^\alpha$ and $\mathcal{W}^\alpha q$ are zero in the chiral ring.

Another important point of these anomaly equations (6.4) and (6.6) is that the ϕ(or $\tilde{\phi}$)-dependence of the gauge kinetic term does not affect the anomaly equation for $R(z)$ (or $\widetilde{R}(z)$), as noted in [9, 10]. In other words, t_k and $\tilde{t}_k$ do not enter in those equations. This is crucial in the analysis in subsequent subsections.

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[5] These anomaly equations were derived also in [27] recently.

6.2 Duality map of $\mathrm{Tr}\,\phi^k \mathcal{W}^\alpha \mathcal{W}_\alpha$ operators

In this subsection, we consider the operators $\mathrm{Tr}\,\phi^k \mathcal{W}^\alpha \mathcal{W}^\alpha$ ($\mathrm{Tr}\,\tilde{\phi}^k \widetilde{\mathcal{W}}^\alpha \widetilde{\mathcal{W}}^\alpha$ in the magnetic theory). We expect from the argument in previous section that the following duality map of the operators are satisfied:

$$\mathrm{Tr}\,\phi^k \mathcal{W}^\alpha \mathcal{W}_\alpha = -\,\mathrm{Tr}\,\tilde{\phi}^k \widetilde{\mathcal{W}}^\alpha \widetilde{\mathcal{W}}_\alpha. \tag{6.8}$$

We will check this relation in the vacuum. It should be noted that in the case without $\bar{Q}\phi Q$ and $m\bar{Q}Q$ terms, the argument in the rest of this section might be invalid. More precisely, we obtain $\langle \mathrm{Tr}\,\phi^k \mathcal{W}^\alpha \mathcal{W}_\alpha \rangle = 0$, as we can see from the anomaly equation for $M(z)$. This is because the flavors remain massless at IR. Therefore, we will consider the full superpotential (5.2) below.

Since $\langle \mathrm{Tr}\,\phi^k \mathcal{W}^\alpha \mathcal{W}_\alpha \rangle = 0$ in the classical vacuum, the classical analysis cannot be non-trivial check of the duality map (6.8). However, they could have non-zero expectation values in the quantum vacuum, as can be seen from the anomaly equations. Indeed, we can relate $R(z)$ to $\widetilde{R}(z)$ by using the generalized Konishi anomaly equations and this will be a non-trivial check of (6.8). Let us see this below.

The generalized Konishi anomaly equations (6.4) and (6.6) for $R(z)$ and $\widetilde{R}(z)$, can be rewritten as

$$R(z)^2 = W'(z)R(z) + \frac{f(z)}{4}, \quad \widetilde{R}(z)^2 = -W'(z)\widetilde{R}(z) + \frac{\tilde{f}(z)}{4}, \tag{6.9}$$

where $f(z)$ and $\tilde{f}(z)$ are the polynomials of degree $n-1$. These equations can be easily solved as

$$R(z) = \frac{1}{2}\left(W'(z) - \sqrt{W'(z)^2 + f(z)}\right), \quad \widetilde{R}(z) = \frac{1}{2}\left(-W'(z) + \sqrt{W'(z)^2 + \tilde{f}(z)}\right), \tag{6.10}$$

where the signs of the square roots have chosen to be consistent with the large z behavior of $R(z)$ and $\widetilde{R}(z)$. From the above forms, we can see that $R(z)$ and $\widetilde{R}(z)$ have cuts in the complex z plane and are, respectively, meromorphic functions on Riemann surfaces Σ and $\tilde{\Sigma}$ of genus $n-1$: $y^2 = W'(z)^2 + f(z)$ and $\tilde{y}^2 = W'(z)^2 + \tilde{f}(z)$. Let us denote by α_i and $\tilde{\alpha}_i$ α-cycles of Σ and $\tilde{\Sigma}$ respectively.

The polynomials $f(z)$ and $\tilde{f}(z)$ are completely fixed [6] by

$$-\frac{1}{64\pi^2}\langle \mathrm{Tr}_{\mathrm{U}(r_i)}\mathcal{W}^{i\alpha}\mathcal{W}^i_\alpha \rangle = \frac{1}{2\pi i}\oint_{\alpha_i} R(z)dz,$$

$$-\frac{1}{64\pi^2}\langle \mathrm{Tr}_{\mathrm{U}(\tilde{r}_i)}\widetilde{\mathcal{W}}^{i\alpha}\widetilde{\mathcal{W}}^i_\alpha \rangle = \frac{1}{2\pi i}\oint_{\tilde{\alpha}_i} \widetilde{R}(z)dz. \tag{6.11}$$

It follows from these equations and (6.2) that $f(z) = \tilde{f}(z)$. Therefore, we obtain $R(z) = -\widetilde{R}(z)$, which implies

$$\langle \mathrm{Tr}\,\phi^k \mathcal{W}^\alpha \mathcal{W}_\alpha \rangle = -\langle \mathrm{Tr}\,\tilde{\phi}^k \widetilde{\mathcal{W}}^\alpha \widetilde{\mathcal{W}}_\alpha \rangle. \tag{6.12}$$

Note that this could be an alternative check of the magnetic gauge kinetic term. Indeed, as we have noted above, $M_i \bar{q}\phi^{k-1}q$ terms and the magnetic gauge kinetic term in the

Lagrangian do not contribute to the anomaly equation for $\widetilde{R}(z)$. What we have assumed in the above argument is that the polynomial part of the magnetic superpotential is $-W(\tilde{\phi})$ and the relations (6.2). However, these follows from that the electric and magnetic superpotentials have the same critical points and that both theories split into the decoupled SQCD theories at low energy. Once we have derived (6.12), we then obtain the following relations by taking derivatives of the partition functions with respect to t_k and $\tilde{t}_k$:

$$\frac{\partial Z}{\partial t_k} \sim \langle \mathrm{Tr}\, \phi^k \mathcal{W}^\alpha \mathcal{W}_\alpha \rangle = -\langle \mathrm{Tr}\, \tilde{\phi}^k \widetilde{\mathcal{W}}^\alpha \widetilde{\mathcal{W}}_\alpha \rangle \sim -\frac{\partial \widetilde{Z}}{\partial \tilde{t}_k} \tag{6.13}$$

Since the duality implies $Z = \widetilde{Z}$ at least for the holomorphic sector, thus we can conclude that the magnetic gauge kinetic term is (5.12).

6.3 Duality map of $\mathrm{Tr}\, \phi^k$ operators

Finally, we analyze the operator relations between $\mathrm{Tr}\, \phi^k$ and $\mathrm{Tr}\, \tilde{\phi}^k$. In the theory with constant τ, it has been known [22] that the duality map can be written as

$$\mathrm{Tr}\, \phi^k = -\mathrm{Tr}\, \tilde{\phi}^k + \frac{k}{\mu^2} \sum_{i=1}^{k-1} M_i \bar{q} \tilde{\phi}^{k-1-i} q + \ldots, \tag{6.14}$$

where ellipsis denotes the constant term. In [22], these relations have been checked by substituting the classical vacuum expectation values. Also, they have been analyzed in [37] by using the generalized Konishi anomaly equations. On general grounds, we can expect that these relations can be deformed by the term with the operators $\mathrm{Tr}\, \tilde{\phi}^k \widetilde{\mathcal{W}}^\alpha \widetilde{\mathcal{W}}_\alpha$ and the terms involving t_k, in the case with $\tau(\phi)$. Let us show below that such terms do not exist by making use of the generalized Konishi anomaly equations.

The third equations of the generalized Konishi anomaly equations (6.4) and (6.6):

$$-\delta_{f'}^f R(z) = \left[M(z)_{f'}^f (z + m_f) \right]_-, \quad -\delta_{f'}^f \widetilde{R}(z) = \left[\widetilde{M}(z)_{f'}^f A(z) \right]_- \tag{6.15}$$

imply that the t_k-dependence cannot enter in $M(z)$ and $\widetilde{M}(z)$, since $R(z)$ and $\widetilde{R}(z)$ are independent of t_k. On the other hand, the second equations of (6.4) and (6.6) are

$$2R(z)T(z) = \left[W'(z)T(z) \right]_- + 32\pi^2 i \left[\tau'(z)R(z) \right]_- + M(z),$$

$$-2R(z)\widetilde{T}(z) = -\left[W'(z)\widetilde{T}(z) \right]_- + 32\pi^2 i \left[\tau'(z)R(z) \right]_- + \left[\widetilde{M}(z)A'(z) \right]_-, \tag{6.16}$$

where we have substituted $R(z) = -\widetilde{R}(z)$. At this stage, we can see that the field-dependent gauge kinetic term *does* affect the quantum chiral ring relation:[6] the second terms in the

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[6]The author thanks Ken Intriligator for a useful comment on this point.

right hand sides denote that $T(z)$ and $\widetilde{T}(z)$ are affected by the gauge kinetic terms. Indeed, in large z, the first equation of (6.16) becomes

$$\left\langle W'(\Phi)\right\rangle - \frac{i}{2}\left\langle \tau'(\Phi)\mathcal{W}^\alpha\mathcal{W}_\alpha\right\rangle + \left\langle \bar{Q}Q\right\rangle = 0, \tag{6.17}$$

and this is the usual F-term equation. In the classical vacua, the second term does not contribute, but it does in the quantum vacua.

Let us consider the effect of the second terms in (6.16). In the constant τ case, by the duality map (6.14), the first equation of (6.16) should reduce to the second equation, as noted above. In the case with $\tau(\phi)$, since the only difference between the second terms in (6.16) is the sign, they do not change the duality map (6.14).

Acknowledgments

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